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SIMPLE MICROFLUIDS*

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Abstract—The basic field equations, jump conditions and constitutive equations of, what we call, 'simple microfluent' media are derived and discussed. These fluids are shown to be a generalization of the Stokesian fluids in which local micro-motions are taken into account. Special cases in which gyrations are small and micro-deformation rates are linear are discussed. The partial differential equations of the constitutively linear theory are obtained.

1. INTRODUCTION

IN A companion paper [1] we gave a nonlinear theory for an elastic solid in which the first stress moments, micro-stress averages and inertial spin play important roles. Elastic solids exhibiting such local effects were named 'simple microelastic materials'. In the present paper we investigate a new class of fluids which exhibit similar micro-effects.

A simple micro-fluid, roughly speaking, is a fluent medium whose properties and behaviour are affected by the local motions of the material particles contained in each of its volume element. A precise definition of such a fluid is given in section 4. The simple micro-fluids possess local inertia. Consequently new principles must be added to the basic principle of continuous media which deals with

(i) conservation of micro-inertia moments
(ii) balance of first stress moments

The theory naturally gives rise to the concept of inertial spin, body moments, micro-stress averages and stress moments which have no counterpart in the classical fluid theories. In these fluids stresses and stress moments are functions of deformation rate tensor, and various micro-deformation rate tensors. Fluids having surface tensions, anistropic fluids, vortex fluids and fluids in which other gyrational effects are important, are conjectured to fall into the domain of simple micro-fluids.

The simple micro-fluids are viscous fluids and in the simplest case of constitutively linear theory these fluids contain 22 viscosity coefficients. Nonlinear Stokesian fluids turn out to be a special class of simple micro-fluids.

In Section 2 we discuss the motion and micro-motions. The gyration tensor, inertial spin and the conservation of micro-inertia and objectivity of micro-deformation rate tensors are derived and discussed. Equations of balance, jump conditions and discussion of entropy production and other relevant thermodynamic concepts occupy Section 3. In Section 4 we give a theory of constitutive equations. The partial differential equations of constitutively linear simple micro-fluids are given in Section 5.

2. MOTION

The motion and the inverse motion of a material point $\mathbf{X}'$, having curvilinear coordinates X'^K, in the undisturbed body $V+S$, are respectively given by the parameter one-to-one mappings

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$$x' = x(X', t), \qquad X' = X(x', t) \tag{2.1}$$

where $\mathbf{x}'$, referred to curvilinear coordinates x'^k, is the spatial point occupied by the material point $\mathbf{X}'$ at time t.

We decompose the motion and the inverse motion as

$$\mathbf{x}' = \mathbf{x}(\mathbf{X}, t) + \boldsymbol{\xi}(\mathbf{X}, \boldsymbol{\Xi}, t), \qquad \mathbf{X}' = \mathbf{X}(x, t) + \boldsymbol{\Xi}(\mathbf{x}, \boldsymbol{\xi}, t) \tag{2.2}$$

where $\boldsymbol{\Xi}$ and $\boldsymbol{\xi}$ are, respectively, the relative position vectors of the material point $\mathbf{X}'$ and its spatial place $\mathbf{x}'$ at time t, with respect to the positions $\mathbf{X}$ and $\mathbf{x}$, respectively, Fig. 1. By selecting

$$\xi^k = \chi_K{}^k \Xi^K, \qquad \chi_K{}^k(\mathbf{X}, t) \equiv \left.\frac{\partial \xi^k}{\partial \Xi^K}\right|_{\Xi^K=0} \tag{2.3}$$

it can be shown that [1] the mass center of a volume element dV in the undisturbed body is carried into the mass center of dv in the deformed body.

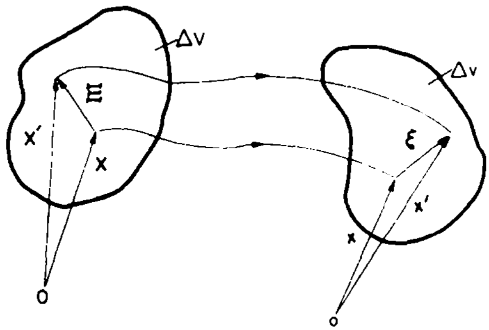

FIG. 1. Undeformed and deformed volume elements.

The inverse micro-motions $\chi^K{}_d(\mathbf{x}, t)$ are defined by

$$\chi^K{}_k \chi^k{}_L = \delta^K_L, \qquad \chi^K{}_k \chi_K{}^l = \delta^l_k. \tag{2.4}$$

Each of the sets in (2.4) is a set of nine linear equations for nine unknowns $\chi^K{}_k$. A unique solution exists in the form

$$\chi_k{}^K = \frac{\text{cofactor}\, \chi_K^k}{J_0} = \frac{1}{2J_0} e^{KLM} e_{klm} \chi_L{}^l \chi_M{}^m \tag{2.5}$$

provided that the jacobian

$$J_0 \equiv \det \chi_K{}^k \neq 0 \tag{2.6}$$

where det ≡ determinant.

For $(2.2)_2$ to be the unique inverse $(2.2)_1$, (2.6) as well as

$$J \equiv \det x^k{}_{,K} \neq 0 \tag{2.7}$$

must be assumed. Condition (2.7) is required for $\mathbf{X}(\mathbf{x}, \mathrm{t})$ to the unique inverse of $\mathbf{x}(\mathbf{X}, \mathrm{t})$.

It is not difficult to see that, dual to (2.3), we have

$$\Xi^K = \chi^K{}_k \xi^k, \qquad \chi^K{}_k \equiv \left.\frac{\partial \Xi^K}{\partial \xi^k}\right|_{\xi^k=0}. \tag{2.8}$$

Next we calculate the velocity $\mathbf{v}'$ and the acceleration $\mathbf{a}'$. These are obtained by taking time rates of (2.2) and using $(2.8)_1$.

$$v'^k = v^k + \nu_l{}^k \xi^l \tag{2.9}$$

$$a'^k = a^k + (\dot{\nu}_l{}^k + \nu_m{}^k \nu_l{}^m)\xi^l \tag{2.10}$$

where

$$\begin{aligned} v^k(\mathbf{x}, t) &\equiv \dot{\mathbf{x}}^k = \left.\frac{\partial x^k}{\partial t}\right|_{\mathbf{X}} \\ a^k(\mathbf{x}, t) &\equiv \frac{\mathrm{D}v^k}{\mathrm{D}t} = \left.\frac{\partial v^k}{\partial t}\right|_{\mathbf{x}} + v^k_{;l}v^l \\ \nu_l{}^k(\mathbf{x}, t) &\equiv \dot{\chi}_K{}^k \chi^K{}_l \\ \dot{\nu}_l{}^k &\equiv \frac{\mathrm{D}}{\mathrm{D}t}(\nu_l{}^k), \qquad \dot{\chi}_K{}^k \equiv \frac{\mathrm{D}}{\mathrm{D}t}(\chi_K{}^k)\,. \end{aligned} \tag{2.11}$$

Here $\mathrm{D}/\mathrm{D}t$ stands for the material derivative and a semicolon indicates covariant partial differentiation with respect to the metric g_{kl} of the coordinates x^k. Note that

$$\dot{\xi}^k = \nu_l{}^k \xi^l\,. \tag{2.12}$$

The tensor $\mathbf{\nu}$ is basic in all of the following developments. We shall name it 'gyration tensor'.

In the sequel we will also need the inertial spin defined by

$$\dot{\sigma}^{kl} = I^{KM}\ddot{\chi}_K{}^k \chi_L{}^l = i^{ml}(\dot{\nu}_m{}^k + \nu_n{}^k \nu_m{}^n) \tag{2.13}$$

where

$$I^{KM} = I^{MK} = \int_{\mathrm{d}V} \rho_0 \Xi^K \Xi^L \mathrm{d}V' \tag{2.14}$$

is a constant material tensor and

$$i^{km} \equiv I^{KM}\chi_K{}^k \chi_M{}^m \tag{2.15}$$

is a corresponding one for the deformed body. We shall name i^{km} as 'micro-inertia moments'.

Theorem 1. *Micro-inertia moments satisfy the following partial differential equations*

$$\frac{\partial i^{km}}{\partial t} + i^{km}{}_{,r}v^r - i^{rm}\nu_r{}^k - i^{kr}\nu_r{}^m = 0\,. \tag{2.16}$$

Proof: Take material derivative of (2.15), i.e.,

$$\frac{\mathrm{D}}{\mathrm{D}t}(i^{km}) = I^{KM}\dot{\chi}_K{}^k \chi_M{}^m + I^{KM}\chi_K{}^k \dot{\chi}_M{}^m\,.$$

From $(2.11)_3$ with the help of (2.4) we solve for

$$\dot{\chi}_K{}^k = \nu_l{}^k \chi_K{}^l\,. \tag{2.17}$$

Using this in the previous equations and rearranging terms we obtain (2.16).

The differential equations (2.16) are the complements to the continuity equations of the hydrodynamics for micro-fluids. We shall name them as the equations of 'conservation of micro-inertia moments'. The integral form of these equations are of course given by (2.15).

Theorem 2. The material derivative of micro-displacement differentials $d\xi^k$ *is given by*

$$\frac{D}{Dt}(d\xi^k) = \nu_l{}^k d\xi^l + \nu_l{}^k{}_{;m}\xi^l dx^m . \tag{2.18}$$

Proof: Take the material derivative of

$$d\xi^k = d\chi_K{}^k \Xi^K + \chi_K{}^k d\Xi^K \tag{2.19}$$

i.e.,

$$\frac{D}{Dt}(d\xi^k) = \frac{D}{Dt}(d\chi_K{}^k)\Xi^K + \frac{D}{Dt}(\chi_K{}^k)d\Xi^K \tag{2.20}$$

but

$$\frac{D}{Dt}(d\chi_K{}^k) = \frac{D}{Dt}(\chi_K{}^k{}_{;l}dx^l) = \frac{D}{Dt}(\chi_K{}^k{}_{;l})dx^l + \chi_K{}^k{}_{;l}v^l{}_{;r}dx^r$$

where we used the well-known result [2, equation 19.1], i.e.,

$$\frac{D}{Dt}(dx^l) = v^l{}_{;m}dx^m . \tag{2.21}$$

The following identity is also needed

$$\frac{D}{Dt}(\chi_K{}^k{}_{;l}) = \left(\frac{D}{Dt}\chi_K{}^k\right)_{;l} - \chi_K{}^k{}_{;m}v^m{}_{;l} . \tag{2.22}$$

Through this and (2.17) we will have

$$\frac{D}{Dt}(d\chi_K{}^k) = (\nu_l{}^k\chi_K{}^l)_{;r}dx^r . \tag{2.23}$$

Using this in (2.20) we obtain (2.18).

Theorem 3. The material derivative of the square of arc length in the deformed body is given by

$$\begin{aligned}\frac{D}{Dt}(ds'^2) = {} & [v_{k;l} + v_{l;k} + (\nu_{rk;l} + \nu_{rl;k})\xi^r]dx^k dx^l \\ & + 2(v_{l;k} + \nu_{lk} + \nu_{rl;k}\xi^r)dx^k d\xi^l + (\nu_{kl} + \nu_{lk})d\xi^k d\xi^l .\end{aligned} \tag{2.24}$$

Proof: Take the material derivative of

$$\begin{aligned} ds'^2 &= g_{kl}(dx^k + d\xi^k)(dx^l + d\xi^l) \\ &= g_{kl}dx^k dx^l + 2g_{kl}dx^k d\xi^l + g_{kl}d\xi^k d\xi^l .\end{aligned}$$

Using (2.21) and (2.18) and rearranging the terms we obtain (2.24).

We define 'deformation rate tensor' **d**, 'micro-deformation rate tensors' **b** and **a** by

$$d_{kl} \equiv \tfrac{1}{2}(v_{k;l} + v_{l;k}) \equiv v_{(k;l)} , \qquad b_{kl} \equiv \nu_{kl} + v_{k;l} , \qquad a_{klm} \equiv \nu_{kl;m} . \tag{2.25}$$

Equation (2.24) can now be expressed as

$$\frac{D}{Dt}(ds'^2) = 2[d_{kl} + a_{r(kl)}\xi^r]dx^k dx^l + 2(b_{lk} + a_{rlk}\xi^r)dx^k d\xi^l + 2[b_{(kl)} - d_{kl}]d\xi^k d\xi^l . \tag{2.26}$$

It is clear that when $\mathbf{d} = \mathbf{b} = 0$ then $a_{rkl} = 0$ and therefore $D(ds'^2)/Dt = 0$. Conversely, for arbitrary dx^k and $d\xi^k$, vanishing $D(ds'^2)/Dt$ implies $\mathbf{d} = \mathbf{b} = 0$. We therefore have proved

Theorem 4. *A necessary and sufficient condition for micro-rigid motion is that* $\mathbf{b}=\mathbf{d}=0$

This theorem replaces the well-known Killing's theorem of differential geometry.

It is well-known that the deformation rate tensor $\mathbf{d}$ is an objective tensor; that is, if $\hat{\mathbf{x}}'(\mathbf{X}', t)$ and $\mathbf{x}'(\mathbf{X}', t)$ are two *objectively* equivalent motions, i.e., referred to rectangular frame of reference

$$\hat{x}'^k = Q_l{}^k x'^l + b'^l \tag{2.27}$$

where $Q^k{}_l$ and b'^l are function of time t alone subject to

$$Q^k{}_l\, Q_m{}^l = Q_l{}^k Q^l{}_m = \delta^k_m, \qquad t' = t - \text{a} \tag{2.28}$$

where a is a constant, then d^k_l transforms as

$$\hat{d}^k{}_l = Q^k{}_r d^r{}_m Q_l{}^m \quad \text{or} \quad \hat{\mathbf{d}} = \mathbf{Q}\mathbf{d}\mathbf{Q}^T . \tag{2.29}$$

Physical interpretation of this is that under the rigid time dependent translation and rotation of the frame of reference and the constant shift of time $\mathbf{d}$ transform like an absolute tensor. We now prove

Theorem 5. *The micro-deformation rate tensor* $\mathbf{b}$ *and* $\mathbf{a}$ *are objective tensors.*

Proof: By putting $x^l + \chi_K{}^l \Xi^K$ and $x^k + \hat{\chi}_K{}^k \Xi^K$ respectively in place of x'^l and $\hat{x}'^k$ in (2.27) we have

$$\hat{x}^k = Q^k{}_r x^r + b^k, \qquad \hat{\chi}_K{}^k = Q^k{}_r \chi_K{}^r . \tag{2.30}$$

From $(2.30)_2$ by differentiation we get

$$\dot{\hat{\chi}}_K{}^k = Q^k{}_r \dot{\chi}_K{}^r + \dot{Q}^k{}_r \chi_K{}^r .$$

Now multiply both sides of this equation by $\hat{\chi}^K{}_l = Q_l{}^m \chi^K{}_m$ and use $(2.4)_2$ and $(2.11)_3$. Hence

$$\hat{\nu}_l{}^k = Q^k{}_r \nu_n{}^r Q_l{}^n + \dot{Q}^k{}_r Q_l{}^r \tag{2.31}$$

but we have, c.f., [2, equation 27.16],

$$\dot{Q}^k{}_r = -Q^n{}_r \hat{v}_{n,}{}^k + Q^k{}_n v_{r,}{}^n .$$

Substituting this into (2.31) we obtain

$$\hat{b}^k{}_l = Q^k{}_r b^r{}_n Q_l{}^n \quad \text{or} \quad \hat{\mathbf{b}} = \mathbf{Q}\mathbf{b}\mathbf{Q}^T \tag{2.32}$$

which proves the part of the theorem concerning $\mathbf{b}$. Objectivity of $\mathbf{a}$ is shown by simply differentiating (2.31) with respect to x^m and using $\partial x^m/\partial \hat{x}^n = Q_m{}^k$. Hence

$$\hat{\nu}_l{}^k{}_{;m} = Q^k{}_r \nu_n{}^r{}_{,p} Q_l{}^n Q_m{}^p \tag{2.33}$$

which completes the proof of the theorem.

3. EQUATIONS OF BALANCE AND ENTROPY

The principle of conservation of mass is expressed by the well-known equation

$$\frac{\partial \rho}{\partial t} + (\rho v^k)_{;k} = 0 \tag{3.1}$$

where ρ is the mass density of the deforming medium. The principle of conservation of micro-inertia moments leads to equations (2.16), i.e.,

$$\frac{\partial i^{km}}{\partial t} + i^{km}{}_{;r}\, v^r - i^{rm} v_r{}^k - i^{kr} v_r{}^m = 0 \,. \tag{3.2}$$

The axioms of local balance of momenta and conservation of energy are expressed by [1].

$$t^{kl}{}_{;k} + \rho(f^l - \dot{v}^l) = 0 \tag{3.3}$$

$$t^{ml} - s^{ml} + \lambda^{klm}{}_{;k} + \rho(l^{lm} - \dot{\sigma}^{lm}) = 0 \tag{3.4}$$

$$\rho\dot{\varepsilon} = t^{kl} v_{l;k} + (s^{kl} - t^{kl}) v_{kl} + \lambda^{klm} v_{ml;k} + q^k_{;k} + \rho h \tag{3.5}$$

valid within the material volume $\mathscr{V}$, and by the jump conditions

$$t^{kl} n_k = t^l_{(\mathbf{n})} \tag{3.6}$$

$$\lambda^{klm} n_k = \lambda^{lm}_{(\mathbf{n})} \tag{3.7}$$

$$q^k n_k = q_{(\mathbf{n})} \tag{3.8}$$

valid on the surface $\mathscr{S}$ of $\mathscr{V}$.

Here

t^{kl} = stress tensor, ρ = mass density
f^l = body force per unit mass
$s^{kl} = s^{lk}$ = micro-stress average
λ^{klm} = the first stress moments
l^{lm} = the first body moment per unit mass
$\dot{\sigma}^{lm}$ = inertial spin
ε = internal energy density per unit mass
q^k = the heat vector
h = the heat source per unit mass
n^k = the exterior unit normal vector to $\mathscr{S}$.

The surface tractions $t^l_{(\mathbf{n})}$, moments $\lambda^{lm}_{(\mathbf{n})}$, heat $q_{(\mathbf{n})}$ and f^l, l^{lm} and h are prescribed quantities or replaced by equivalent information.

Axiom. A simple micro-fluid is assumed to possess an internal energy function ε *which depends solely on entropy* η, *specific volume* $1/\rho$ *and the micro-inertia moments* i^{km}

$$\varepsilon = \varepsilon\left(\eta, \frac{1}{\rho}, i^{km}\right). \tag{3.9}$$

We assume that ε is continuously differentiable with respect to its arguments and define the thermodynamic tensions by

$$\theta \equiv \frac{\partial \varepsilon}{\partial \eta}\bigg|_{\rho^{-1}, \mathbf{i}}, \qquad \pi \equiv -\frac{\partial \varepsilon}{\partial \rho^{-1}}\bigg|_{\eta, \mathbf{i}}, \qquad \pi_{km} \equiv \frac{\partial \varepsilon}{\partial i^{km}}\bigg|_{\rho^{-1}, \eta}. \tag{3.10}$$

Here θ is called the 'thermodynamic temperature', π the 'thermodynamic pressure' and π_{km} the 'thermodynamic micro-pressure tensor'.

Since $\eta = \eta(\mathbf{x}, t)$, $\rho = \rho(\mathbf{x}, t)$ and $\mathbf{i} = \mathbf{i}(\mathbf{x}, t)$ from (3.9) by differentiation and using (3.1) and (3.2), it follows that

$$\dot{\varepsilon} = \theta\dot{\eta} - \frac{\pi}{\rho} v^k_{;k} + 2\pi_{km} i^{rm} v_r{}^k \,. \tag{3.11}$$

We decompose the stress tensors **t** and **s** into two parts

$$t_l^{\ k} = -\bar{p}\delta^k_{\ l} + \bar{t}^k_{\ l}\,, \qquad s^k_{\ l} = -\bar{p}\delta^k_{\ l} + \bar{s}^k_{\ l} \tag{3.12}$$

with $\bar{p}$ representing hydrostatic pressure.*

Upon replacing $\dot{\varepsilon}$ in (3.5) by (3.11) and using (3.12) we obtain the differential equation of entropy production

$$\rho\theta\dot{\eta} = (\pi - \bar{p})v^k_{;k} + \bar{t}_l^{\ k}v^l_{;k} + (\bar{s}^k_{\ l} - \bar{t}^k_{\ l} - \tau^k_{\ l})\nu_k^{\ l} + \lambda^{k\ m}_{\ l}\nu_m^{\ l}{}_{;k} + q^k_{;k} + \rho h \tag{3.13}$$

where

$$\tau^k_l \equiv 2\rho i^{km}\pi_{lm} \tag{3.14}$$

will be named the *thermodynamic micro-stress tensor*. From the differential equation (3.13) of entropy it is clear that the following dissipative forces contribute to the time rate of change of entropy

(a) the difference between the mechanical and thermodynamic pressures
(b) the stress power
(c) the difference of micro and thermodynamic stresses from the stress
(d) the stress moments
(e) the heat input and the heat sources.

It is interesting to note that the micro-fluid with no rigid structure possesses a reversible thermodynamic stress whose energy must be subtracted from the stress energies in calculating the entropy production. This reversible energy is not encountered in the classical Stokesian fluids.

If we write (3.13) in the equivalent form

$$\rho\dot{\eta} - (q^k/\theta)_{;k} = \Delta + \frac{\rho h}{\theta} \tag{3.15}$$

where

$$\theta\Delta \equiv (\pi - \bar{p})v^k_{;k} + \bar{t}^k_{\ l}v^l_{;k} + (\bar{s}^k_{\ l} - \bar{t}^k_{\ l} - \tau^k_{\ l})\nu_k^{\ l} + \lambda^{k\ m}_{\ l}\nu_m^{\ l}{}_{;k} + q^k(\log\theta)_{;k}\,. \tag{3.16}$$

By integration of (3.15) over the volume we obtain

$$\dot{H} - \oint_{\mathscr{S}} \frac{q^k}{\theta}\,\mathrm{d}a_k = \int_{\mathscr{V}} \left(\Delta + \frac{\rho h}{\theta}\right)\mathrm{d}v \tag{3.17}$$

where

$$H \equiv \int_{\mathscr{V}} \rho\eta\,\mathrm{d}v \tag{3.18}$$

is the total entropy. The Clausius–Duhem inequality

$$\dot{H} - \oint_{\mathscr{S}} \frac{q^k}{\theta}\,\mathrm{d}a_k \geqslant 0 \tag{3.19}$$

is obtained from (3.17) for $\theta \geqslant 0$ if

$$\theta\Delta \geqslant 0\,, \qquad \rho h \geqslant 0\,. \tag{3.20}$$

* Since $\bar{p}$ is somewhat arbitrary there is no reason to take different $\bar{p}$, for **t** and **s** thus introducing two different pressures. Single pressure is also indicated through the dependence of ε on single density ρ in (3.9)

In accordance with the tradition we only admit those values of $\theta \geqslant 0$; then (3.19) implies (3.17). Further progress in dealing with $(3.20)_1$ requires additional assumptions requiring independent non negative character of various dissipative energies, e.g.,

$$q^k(\log\theta)_{,k} \geqslant 0\,, \quad \text{etc.}$$

Since our intention, here, is not to examine closely the foundations of the thermodynamics or to dwell into the thermal problems, we leave the subject matter of entropy at this point.

4. CONSTITUTIVE EQUATIONS

Definition

A fluent medium is called a simple micro-fluid if it possesses constitutive equations of the form

$$\begin{aligned} t_{kl} &= f_{kl}(v_{p;q}\,,\ \nu_{pq}\,,\ \nu_{pq;r}) \\ s_{kl} &= g_{kl}(v_{p;q}\,,\ \nu_{pq}\,,\ \nu_{pq;r}) \\ \lambda_{klm} &= h_{klm}(v_{p;q}\,,\ \nu_{pq}\,,\ \nu_{pq;r}) \end{aligned} \tag{4.1}$$

subject to spatial and material objectivity and

$$t_{kl} = s_{kl} = -\pi g_{kl}\,, \qquad \lambda_{klm} = 0 \tag{4.2}$$

when $\mathbf{d} = \mathbf{b} = 0$.

According to the principle of objectivity (4.1) must be form-invariant in any two objectively equivalent motions $\hat{\mathbf{x}}'$ and $\mathbf{x}'$ related to each other by (2.27). This imposes conditions on the forms of the functions f_{kl}, g_{kl} and h_{klm}. In order to apply the principle of objectivity one takes both frames $\hat{\mathbf{x}}'$ and $\mathbf{x}'$ rectangular and connected by (2.27) or equivalently (2.30) for $\hat{\mathbf{x}}$ and $\hat{\boldsymbol{\chi}}$. In the frame $\hat{\mathbf{x}}'$ we must have

$$\hat{t}_{kl} = f_{kl}(\hat{v}_{p,q}\,,\ \hat{\nu}_{pq}\,,\ \hat{\nu}_{pq,r}) \tag{4.3}$$

and similar expressions for $\hat{s}_{kl}$ and $\hat{\lambda}_{klm}$. We have

$$\begin{aligned} \hat{t}^k{}_l &= Q^k{}_m t^m{}_n Q_l{}^n\,, & \hat{v}^k{}_{,l} &= Q^k{}_r v^r{}_{,n} Q_l{}^n + \dot{Q}^k{}_r Q_l{}^r \\ \hat{\nu}_l{}^k &= Q^k{}_r \nu_n^r Q_l{}^n + \dot{Q}^k{}_r Q_l{}^r\,, & \hat{\nu}_l{}^k{}_{,m} &= Q^k{}_r \nu_n{}^r{}_{,p} Q_l{}^n Q_m{}^p \end{aligned} \tag{4.4}$$

c.f. [2, equation 48.10], (2.31) and (2.33). Thus, (4.1), (4.3) and (4.4) imply that

$$\begin{aligned} &\mathbf{Q}\mathbf{f}(v^k{}_{,l}\,,\ \nu_l{}^k,\ \nu_l{}^k{}_{,m})\mathbf{Q}^T \\ &\quad = \mathbf{f}(Q^k{}_r v^r{}_{,n} Q_l{}^n + \dot{Q}^k{}_r Q_l{}^r\,,\ Q^k{}_r \nu_n{}^r Q_l{}^n + \dot{Q}^k{}_r Q_l{}^r\,,\ Q^k{}_r \nu_n{}^r{}_{,p} Q_l{}^n Q_m{}^p) \end{aligned} \tag{4.5}$$

valid for all v_n^r, ν_n^r, $\nu_n{}^r{}_{,p}$ and all orthogonal tensors $\mathbf{Q}$. Now we can always select $\mathbf{Q} = \mathbf{I}$ and $\dot{\mathbf{Q}}$ equal to an antisymmetric tensor transformation given a priori. Selecting

$$Q^k{}_r = \delta^k_r\,, \qquad \dot{Q}^k{}_r = w_r{}^k = \tfrac{1}{2}(v_{r,}{}^k - v^k{}_{,r})$$

we see that (4.5) reduces to the first of the following equations

$$\mathbf{t} = \mathbf{f}(\mathbf{d},\ \mathbf{b}-\mathbf{d},\ \mathbf{a})\,, \qquad \mathbf{s} = \mathbf{g}(\mathbf{d},\ \mathbf{b}-\mathbf{d},\ \mathbf{a})\,, \qquad \boldsymbol{\lambda} = \mathbf{h}(\mathbf{d},\ \mathbf{b}-\mathbf{d},\ \mathbf{a})\,. \tag{4.6}$$

Equations for $\mathbf{s}$ and $\boldsymbol{\lambda}$, here, are obtained by the same procedure. Arguments of $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$ are now objective tensors and these functions are subject to

$$\begin{aligned}
&\mathbf{f}(\mathbf{QdQ}^T,\ \mathbf{Q(b-d)Q}^T,\ \mathbf{QaQ}^T\mathbf{Q}^T)=\mathbf{Qf(d,\ b-d,\ a)Q}^T\\
&\mathbf{g}(\mathbf{QdQ}^T,\ \mathbf{Q(b-d)Q}^T,\ \mathbf{QaQ}^T\mathbf{Q}^T)=\mathbf{Qg(d,\ b-d,\ a)Q}^T\\
&\mathbf{h}(\mathbf{QdQ}^T,\ \mathbf{Q(b-d)Q}^T,\ \mathbf{QaQ}^T\mathbf{Q}^T)=\mathbf{Qh(d,\ b-d,\ a)Q}^T\mathbf{Q}^T
\end{aligned} \tag{4.7}$$

where ambiguous expressions such as $\mathbf{Q}\,\mathbf{a}\,\mathbf{Q}^T\,\mathbf{Q}^T$ are understood to represent the transformation of the type (2.30). Equations of (4.5) are valid for all orthogonal tensors $\mathbf{Q}$. Selecting $\mathbf{Q}=-\mathbf{I}$, (4.7) gives

$$\begin{aligned}
&\mathbf{f}(\mathbf{d},\ \mathbf{b}-\mathbf{d},\ -\mathbf{a})=\mathbf{f}(\mathbf{d},\ \mathbf{b}-\mathbf{d},\ \mathbf{a})\\
&\mathbf{g}(\mathbf{d},\ \mathbf{b}-\mathbf{d},\ -\mathbf{a})=\mathbf{g}(\mathbf{d},\ \mathbf{b}-\mathbf{d},\ \mathbf{a})\\
&\mathbf{h}(\mathbf{d},\ \mathbf{b}-\mathbf{d},\ -\mathbf{a})=-\mathbf{h}(\mathbf{d},\ \mathbf{b}-\mathbf{d},\ \mathbf{a})\,.
\end{aligned} \tag{4.8}$$

Consequently $\mathbf{f}$ and $\mathbf{g}$ are *even* in $\mathbf{a}$ and $\mathbf{h}$ is *odd* in $\mathbf{a}$. Therefore

$$\mathbf{h}(\mathbf{d},\ \mathbf{b}-\mathbf{d},\ 0)=0 \tag{4.9}$$

and we proved the theorems

*Theorem 6. A second order objective tensor can only be an even tensor function of odd order tensors. A third order objective tensor can only be an odd function of any objective third order tensor**

Corollary. Stress moments vanish with vanishing micro-deformation rate tensor $\mathbf{a}$.

Theorem 7. The constitutive equations of simple micro-fluids are equivalent to†

$$t^k{}_l=f^k{}_l(\mathbf{d},\ \mathbf{b}-\mathbf{d},\ \mathbf{a})\,,\qquad s^k_l=g^k{}_l(\mathbf{d},\ \mathbf{b}-\mathbf{d},\ \mathbf{a})\,,\qquad \lambda^k_{lm}=h^k{}_{lm}(\mathbf{d},\ \mathbf{b}-\mathbf{d},\ \mathbf{a}) \tag{4.10}$$

where $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ are subject to (4.7) to (4.9) and

$$f^k{}_l(0,\ 0,\ 0)=-\pi\delta^k_l\,,\qquad g^k{}_l(0,\ 0,\ 0)=-\pi\delta^k_l\,,\qquad h^k{}_{lm}(0,\ 0,\ 0)=0\,. \tag{4.11}$$

Equations (4.7) impose conditions on the forms of $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$. These conditions are similar to conditions of isotropic tensors. Since third order tensors are involved, the determination of the complete invariants of tensors $\mathbf{d}$, $\mathbf{b}$, and $\mathbf{a}$ poses a tedious and lengthy study which presently is not available or we could not locate any work on this subject. However, by the fact that the micro-motions represented by the tensors $\mathbf{b}-\mathbf{d}$ and $\mathbf{a}$ are generally small we proceed to obtain power series representation of the constitutive equations in $\mathbf{b}-\mathbf{d}$ and $\mathbf{a}$ stopping at the first order terms. Thus we write

$$\begin{aligned}
&t^{kl}=\underset{0}{f}{}^{kl}(\mathbf{d},\ \mathbf{b}-\mathbf{d},\ \mathbf{b}^T-\mathbf{d})+O(\mathbf{a}^2)\\
&s^{kl}=\underset{0}{g}{}^{kl}(\mathbf{d},\ \mathbf{b}-\mathbf{d},\ \mathbf{b}^T-\mathbf{d})+O(\mathbf{a}^2)\\
&\lambda^{klm}=\underset{0}{h}{}^{klm}(\mathbf{d},\ \mathbf{b}-\mathbf{d},\ \mathbf{b}^T-\mathbf{d})+\underset{1}{h}{}^{klmrst}(\mathbf{d},\ \mathbf{b}-\mathbf{d},\ \mathbf{a})a_{rst}+O(\mathbf{a}^3)\,.
\end{aligned} \tag{4.12}$$

* A part of this theorem is usually attributed to P. Curie as the Curie Principle. In the references made [3], I have been able to find the following vague statement: "*Autrement dit, certains éléments de symétrie peuvent coexister avec certains phénomènes, mais ils ne sont pas nécessaires. Ce qui est nécessaire, c'est que certains éléments de symétrie n'existent pas. C'est la dissymétrie qui crée le phénomène*". Several pages of long explanations and examples that follow this statement confuses the matter further by mixing material symmetry with what we now call spatial objectivity.

† We note that $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{h}$ may be taken functions of $\mathbf{d}$, $\mathbf{b}$, and $\mathbf{a}$ rather than $\mathbf{d}$, $\mathbf{b}$–$\mathbf{d}$, $\mathbf{a}$ if we wish. The form (4.10) is convenient for the purpose of linearization about $\mathbf{b}$–$\mathbf{d}$ and $\mathbf{a}$.

The inclusion of $\mathbf{b}^T-\mathbf{d}$, the transpose of $\mathbf{b}-\mathbf{d}$, into the arguments of these functions is necessitated for the purpose of making these functions isotropic functions since $\mathbf{t}$, $\mathbf{b}$ and $\boldsymbol{\lambda}$ are *not*, in general, symmetric tensors. When a further assumption is made to the fact that the tensor functions $\underset{0}{\mathbf{f}}$, $\underset{0}{\mathbf{g}}$, $\underset{0}{\mathbf{h}}$ and $\underset{1}{\mathbf{h}}$ are polynomials in matrices $\mathbf{d}$, $\mathbf{b}-\mathbf{d}$ and $\mathbf{b}^T-\mathbf{d}$ we can express them in finite number of terms. Using the results of [4] one can show that, c.f., [5, equation A.1] $\mathbf{f}$ and $\mathbf{g}$ are expressible as polynomials each having 85 terms. In order not to crowd the present paper with such lengthy expressions, as stated above, we confine our attention to the polynomials linear in $\mathbf{b}-\mathbf{d}$ and $\mathbf{b}^T-\mathbf{d}$. Hence

$$\begin{aligned}\mathbf{t}=&\alpha_0\mathbf{I}+\alpha_1\mathbf{d}+\alpha_2\mathbf{d}^2+\alpha_3(\mathbf{b}-\mathbf{d})+\alpha_4(\mathbf{b}^T-\mathbf{d})+\alpha_5\mathbf{d}(\mathbf{b}-\mathbf{d})+\alpha_6(\mathbf{b}-\mathbf{d})\mathbf{d}\\&+\alpha_7\mathbf{d}(\mathbf{b}^T-\mathbf{d})+\alpha_8(\mathbf{b}^T-\mathbf{d})\mathbf{d}+\alpha_9\mathbf{d}^2(\mathbf{b}-\mathbf{d})+\alpha_{10}(\mathbf{b}-\mathbf{d})\mathbf{d}^2\\&+\alpha_{11}\mathbf{d}^2(\mathbf{b}^T-\mathbf{d})+\alpha_{12}(\mathbf{b}^T-\mathbf{d})\mathbf{d}^2+\alpha_{13}\mathbf{d}(\mathbf{b}-\mathbf{d})\mathbf{d}^2+\alpha_{14}\mathbf{d}(\mathbf{b}^T-\mathbf{d})\mathbf{d}^2\,.\end{aligned}\tag{4.13}$$

An identical expression with α_κ replaced by β_κ', ($\kappa=0, 1, \ldots, 14$) is valid for $\mathbf{s}$. The coefficients α_κ and β_κ', for ($\kappa=0, 1, 2$) are polynomials of the following six invariants

$$\begin{matrix}\operatorname{tr}\mathbf{d}\,, & \operatorname{tr}\mathbf{d}^2\,, & \operatorname{tr}\mathbf{d}^3\\ \operatorname{tr}(\mathbf{b}-\mathbf{d})\,, & \operatorname{tr}(\mathbf{b}-\mathbf{d})\mathbf{d}\,, & \operatorname{tr}(\mathbf{b}-\mathbf{d})\mathbf{d}^2\end{matrix}\tag{4.14}$$

being linear in the last three, and α_κ and β_κ', ($\kappa=3, 4, \ldots, 14$) are functions of the first three invariants $\operatorname{tr}\mathbf{d}$, $\operatorname{tr}\mathbf{d}^2$, and $\operatorname{tr}\mathbf{d}^3$ alone, i.e.,

$$\begin{aligned}&\alpha_\kappa=[\alpha_{\kappa 0}+\alpha_{\kappa 1}\operatorname{tr}(\mathbf{b}-\mathbf{d})+\alpha_{\kappa 2}\operatorname{tr}(\mathbf{b}-\mathbf{d})\mathbf{d}+\alpha_{\kappa 3}\operatorname{tr}(\mathbf{b}-\mathbf{d})\mathbf{d}^2]\\&\alpha_{\kappa\lambda}=\alpha_{\kappa\lambda}(\operatorname{tr}\mathbf{d}\,,\ \operatorname{tr}\mathbf{d}^2\,,\ \operatorname{tr}\mathbf{d}^3)\,,\qquad \kappa=0, 1, 2\\&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\lambda=0, 1, 2, 3\,.\end{aligned}\tag{4.15}$$

We now use the condition of symmetry for $\mathbf{s}$ to reduce it further. In this case $\mathbf{s}=\mathbf{s}^T$ and we obtain

$$\begin{aligned}\mathbf{s}=&\beta_0\mathbf{I}+\beta_1\mathbf{d}+\beta_2\mathbf{d}^2+\beta_3(\mathbf{b}+\mathbf{b}^T-2\mathbf{d})+\beta_4(\mathbf{db}+\mathbf{b}^T\mathbf{d}-2\mathbf{d}^2)\\&+\beta_5(\mathbf{bd}+\mathbf{db}^T-2\mathbf{d}^2)+\beta_6(\mathbf{d}^2\mathbf{b}+\mathbf{b}^T\mathbf{d}^2-2\mathbf{d}^3)+\beta_7(\mathbf{bd}^2+\mathbf{d}^2\mathbf{b}-2\mathbf{d}^3)\\&+\beta_8(\mathbf{dbd}^2+\mathbf{d}^2\mathbf{b}^T\mathbf{d}-2\mathbf{d}^4)+\beta_9(\mathbf{db}^T\mathbf{d}^2+\mathbf{d}^2\mathbf{bd}-2\mathbf{d}^4)\end{aligned}\tag{4.16}$$

where β_0, β_1 and β_2 have the same functional form as in (4.15) with the coefficients, $\alpha_{\kappa\lambda}$, replaced by $\beta_{\kappa\lambda}$ and $\beta_4, \ldots, \beta_9$ are polynomials in the first three invariants listed in (4.14).

To determine the polynomial form of $\boldsymbol{\lambda}$ we first recall (4.9) which implies that $\underset{0}{h}^{klm}=0$. Now $\underset{1}{h}^{klmrst}$ is an isotropic tensor of six order so that upon substituting the known expression of a six order isotropic tensor we get

$$\begin{aligned}\lambda^k{}_{lm}=&(\gamma_1 a_m{}^r{}_r+\gamma_2 a^r{}_{mr}+\gamma_3 a^r{}_{rm})\delta_l^k+(\gamma_4 a_l{}^r{}_r+\gamma_5 a^r{}_{lr}+\gamma_6 a^r{}_{rl})\delta_m^k\\&+(\gamma_7 a^{kr}{}_r+\gamma_8 a^{rk}{}_r+\gamma_9 a^r{}_r{}^k)g_{lm}+\gamma_{10}a^k{}_{lm}+\gamma_{11}a^k{}_{ml}+\gamma_{12}a_l{}^k{}_m+\gamma_{13}a_m{}^k{}_r+\gamma_{14}a_{lm}{}^k+\gamma_{15}a_{ml}{}^k\end{aligned}\tag{4.17}$$

where

$$\gamma_\kappa=\gamma_\kappa(\mathbf{d}\,,\ \mathbf{b}-\mathbf{d}\,,\ \mathbf{b}^T-\mathbf{d})\,,\qquad(\kappa=1, 2, \ldots, 15)\,.\tag{4.18}$$

Since γ_κ are also scalar invariants under all orthogonal transformation $\mathbf{Q}$, when these functions are analytic in their arguments, then, in the micro-linear case under consideration, they are expressible as polynomials in the first three of the six invariants listed in (4.14)

$$\gamma_\kappa=\gamma_\kappa(\operatorname{tr}\mathbf{d}\,,\ \operatorname{tr}\mathbf{d}^2\,,\ \operatorname{tr}\mathbf{d}^3)\,,\qquad(\kappa=1, 2, \ldots, 15)\,.\tag{4.19}$$

The conditions (4.11) are satisfied by taking

$$\begin{aligned}&\alpha_0=-\pi+\alpha(\mathbf{d},\ \mathbf{b}-\mathbf{d},\ \mathbf{b}^T-\mathbf{d})\\&\beta_0=-\pi+\beta(\mathbf{d},\ \mathbf{b}-\mathbf{d},\ \mathbf{b}^T-\mathbf{d})\\&\gamma_\kappa(0,\ 0,\ 0)=\alpha(0,\ 0,\ 0)=\beta(0,\ 0,\ 0)=0\,,\qquad(\kappa=1,\ 2,\ \dots,\ 15)\end{aligned}\tag{4.20}$$

where α and β are functions of the six invariants listed in (4.14).

Theorem 8. *All Stokesian fluids are included in the class of simple micro-fluids represented by the constitutive equations* (4.13), (4.15) *and* (4.16) *subject to* (4.20)
Proof: Take all $\alpha_\kappa=\beta_\kappa=0$ for $\kappa\geqslant 3$ and $\gamma_\kappa=0$ for all κ; then (4.13), (4.15) and (4.16) reduce to

$$\mathbf{t}=(-\pi+\alpha)\mathbf{I}+\alpha_1\mathbf{d}+\alpha_2\mathbf{d}^2\,,\qquad \mathbf{s}=(-\pi+\beta)\mathbf{I}+\beta_1\mathbf{d}+\beta_2\mathbf{d}^2\tag{4.21}$$

where α, α_1, α_2, β, β_1, and β_2 are now to be considered as function of the following three invariants

$$\operatorname{tr}\mathbf{d}\,,\qquad \operatorname{tr}\mathbf{d}^2\,,\qquad \operatorname{tr}\mathbf{d}^3\tag{4.22}$$

or equivalently

$$I_d\,,\qquad II_d\,,\qquad III_d\tag{4.23}$$

as defined in [2, Section 48]. If we now select $\alpha=\beta$, $\alpha_1=\beta_1$ and $\alpha_2=\beta_2$ then we get $\mathbf{t}=\mathbf{s}$. Further when i^{ml} and $\mathbf{l}$ are taken zero then $\dot{\boldsymbol{\sigma}}=0$ and the balance equations (3.2), (3.4) are automatically satisfied and (3.1), (3.3) and (3.5) reduce to those valid for Stokesian fluids.

Theorem 9. *For special types of body and surface moments and for* $\mathbf{d}=\mathbf{b}$ *all motions of simple micro-fluids coincide with those of the Stokesian fluids*
Proof: By taking $\mathbf{d}=\mathbf{b}$ the constitutive equations for stresses reduce to $(4.21)_1$ and $(4.21)_2$. Stress moments $\boldsymbol{\lambda}$ are fully determined through (4.17) by putting

$$a_{klm}=-w_{kl;m}=d_{kl;m}-v_{k;lm}\tag{4.24}$$

which is the result of $\mathbf{d}=\mathbf{b}$. Thus the balance equations (3.4) gives a special distribution for $\mathbf{l}$ and the boundary conditions (3.7) a special surface moment distribution $\lambda^{lm}_{(\mathbf{n})}$. In this case with $\alpha=\beta$, $\alpha_1=\beta_1$ and $\alpha_2=\beta_2$ we are left with the basic equations and boundary conditions of Stokesian fluids.

The constitutive equations (4.13), (4.16) and (4.17) may be used as a master set from which many special and approximate theories may be extracted.* Here we give only the linear theory.

The linear theory. Expanding the constitutive coefficients α_κ, β_κ and γ_κ into power series of their arguments and retaining only the linear terms in $\mathbf{d}$ and $\mathbf{b}$ we obtain

$$\begin{aligned}&\mathbf{t}=[-\pi+\lambda_v\operatorname{tr}\mathbf{d}+\lambda_0\operatorname{tr}(\mathbf{b}-\mathbf{d})]\mathbf{I}+2\mu_v\mathbf{d}+2\mu_0(\mathbf{b}-\mathbf{d})+2\mu_1(\mathbf{b}^T-\mathbf{d})\\&\mathbf{s}=[-\pi+\eta_v\operatorname{tr}\mathbf{d}+\eta_0\operatorname{tr}(\mathbf{b}-\mathbf{d})]\mathbf{I}+2\zeta_v\mathbf{d}+\zeta_1(\mathbf{b}+\mathbf{b}^T-2\mathbf{d})\end{aligned}\tag{4.25}$$

where λ_v, λ_0, μ_v, μ_0, μ_1, η_v, η_0, ζ_v, and ζ_1 are viscosity coefficients. They are, in general, functions of temperature. In order to have the Stokesian fluids included in the linear theory we also take

$$\lambda_v=\eta_v\,,\qquad \mu_v=\zeta_v\,.\tag{4.26}$$

* Further generalizations of the present theory is possible and is presently under consideration.

Thus *the linear theory of simple micro-fluids introduces five additional viscosities into the stress constitutive equations.* The form of the constitutive equations for stress moments is identical to (4.16) except that γ_κ are now constants (or, in general, functions of temperature alone). *Including the gyroviscosities* γ_κ *the total number of viscosity coefficients is* 22.

5. PARTIAL DIFFERENTIAL EQUATIONS OF MOTION

The partial differential equations of motion are obtained by adjoining the equations of conservation of mass (3.1) and conservation of micro-inertia moments (3.2) to those obtained by substituting the constitutive equations (4.13), (4.16) and (4.17) into the equations of balance (3.3) and (3.4). Here we give only the result for the constitutively linear theory.

$$\frac{\partial \rho}{\partial t} + (\rho v^k)_{;k} = 0 \tag{5.1}$$

$$\frac{\partial i^{km}}{\partial t} + i^{km}{}_{;r} v^r - i^{rm} v_r{}^k - i^{rk} v_r{}^m = 0 \tag{5.2}$$

$$\begin{aligned} -\pi_{,l} + (\lambda_v + \mu_v + \mu_0 - \mu_1) v^k{}_{;lk} + (\mu_v - \mu_0 + \mu_1) v_{l;}{}^k{}_k \\ + \lambda_0 v^k{}_{k;l} + 2\mu_0 v^k{}_{l;k} + 2\mu_1 v_l{}^k{}_{;k} + \rho(f_l - \dot{v}_l) = 0 \end{aligned} \tag{5.3}$$

$$\begin{aligned} (\mu_0 - \mu_1)(v^k{}_{;l} - v_{l;}{}^k) + (\lambda_0 - \eta_0) v_m{}^m \delta^k{}_l + (2\mu_0 - \zeta_1) v^k{}_l + (2\mu_1 - \zeta_1) v_l{}^k + (\gamma_1 + \gamma_{13}) v^{km}{}_{;ml} \\ + (\gamma_2 + \gamma_{11}) v^{mk}{}_{;ml} + (\gamma_3 + \gamma_6) v^m{}_{m;l}{}^k + (\gamma_4 + \gamma_{12}) v_l{}^m{}_{;m}{}^k + (\gamma_5 + \gamma_{10}) v^m{}_{l;}{}^k{}_m + \gamma_{14} v_l{}^k{}_;{}^m{}_m \\ + \gamma_{15} v^k{}_{l;}{}^m{}_m + (\gamma_7 v^{mn}{}_{;nm} + \gamma_8 v^{nm}{}_{;nm} + \gamma_9 v^m{}_{m;}{}^n{}_n) \delta^k{}_l + \rho(l_l{}^k - \dot{\sigma}_l{}^k) = 0 . \end{aligned} \tag{5.4}$$

We have $1+6+3+9=19$ equations to determine the 19 unknowns ρ, $i^{km} = i^{mk}$, $v_l{}^k$ and v^k since the body force f_l and the body moment $l_l{}^k$ are supposed to be given and $\dot{\sigma}_l{}^k$ according to (2.13) is expressible in terms of i^{km} and $v_l{}^k$, i.e.,

$$\dot{\sigma}_l{}^k = i^{mk}(\dot{v}_{ml} + v_{nl} v_m{}^n) . \tag{5.5}$$

Under appropriate boundary conditions, e.g., equations (3.6) and (3.7) and initial conditions the complete behaviour of constitutively linear theory of simple micro-fluids should be derivable from the foregoing nonlinear partial differential equations. As the initial conditions one may suggest the initial value problem of Cauchy namely prescribing the initial values of ρ, i^{km}, $v_l{}^k$ and v^k at time $t=0$. The final judgment on whether a boundary and initial value problem is well posed requires the proof of existence and uniqueness theorems. The difficulties encountered on this question for the simple case of Navier–Stokes flows are well-known.

Finally we note that by setting $\lambda_0 = \mu_0 = \mu_1 = \eta_0 = \zeta_1 = l^{km} = i^{km} = \gamma_\kappa = 0$ (5.3) reduce to Navier–Stokes equations; (5.1) remains valid and equations (5.2) and (5.4) reduce to identities $0=0$. This is but one more verification of the fact that Navier–Stokes equations are special cases of those of the simple micro-fluids.

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Résumé—Dans cette étude on détermine et on discute les équations fondamentales de champ, les conditions de passage et les équations d'état de ce qu'on peut appeler les milieux à 'micro-écoulements' simples. On montre que ces milieux correspondent à une généralisation des fluides de Stokes dans lesquels on fait intervenir des 'micro-déplacements' localisés. On présente la discussion de certains cas particuliers dans lesquels la gyrations est faible et où les taux de micro-déformation sont linéaires. On obtient ainsi les équations différentielles aux dérivées partielles de la théorie linéaire de constitution.

Zusammenfassung—Die grundlegenden Feldgleichungen, Sprungverhältnisse und Aufbaugleichungen von was wir „einfache mikro-flüssige" Mittel nennen, werden abgeleitet und besprochen. Diese Flüssigkeiten werden als eine Verallgemeinerung der Stokes–Flüssigkeiten gezeigt, bei denen örtliche Mikro–Bewegungen in die Berechnung einbezogen werden. Sonderfälle bei denen es kleine Wirbel und lineare Mikro–Verformungssätze gibt, werden besprochen. Die teilweise abgeleiteten Gleichungen der aufbauenden linearen Theorie werden erzielt.

Sommario—Si derivano e si esaminano le equazioni fondamentali di campo, cause di errore ed equazioni essenziali relative a quelle che vengono dette sostanze 'semplici micro–fluenti'. Si evidenzia essere questi fluidi una generalizzazione dei fluidi di Stokes tenendo conto di micro–movimenti locali. Si esaminano casi particolari di limitata rotazione e di valori lineari delle microzdeformazioni. Si ottengono le equazioni differenziali abbreviate della teoria lineare fondamentale.

Абстракт—Выводятся и дискуссируются уравнения поля, скачковые условия и конститутивные уравнения для так называемой «микро-текучей» среды.

Показано, что такие жидкости являются обобщением Стокезиевых жидкостей, в которых обращено особое внимание на микро-движение. Дискуссируются особые случаи, в которых вращение мало и скорости микро-деформаций являются линейными. Получены дифференциальные уравнения в частных производных к конститутивной теории.